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ON THE GROUP OF AUTOMORPHISMS OF AFFINE ALGEBRAIC GROUPS

BY

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ABSTRACT. We study the conservativeness property of affine algebraic groups over an algebraically closed field of characteristic 0 and of their group of automorphisms. We obtain a certain decomposition of affine algebraic groups, and this, together with the result of Hochschild and Mostow, becomes a major tool in our study of the conservativeness property of the group of automorphisms.

1. **Introduction.** Let G be an affine algebraic group over a field F, with Hopf algebra $\mathcal{Q}(G)$ of polynomial functions on G, in the sense of [2] and let $W(G)$ denote the group of all affine algebraic group automorphisms of G. Then $\mathcal{Q}(G)$ may be viewed as a right $W(G)$-module, with $W(G)$ acting by composition $f \to f \circ \alpha$ on $\mathcal{Q}(G)$.

We recall, from [3], that G is said to be *conservative* if $\mathcal{Q}(G)$ is locally finite as a $W(G)$-module. As is shown in [3], the conservativeness of G characterizes the existence of a suitable affine algebraic group structure on $W(G)$ and the obstruction to the conservativeness of a connected G is realized as the presence of certain central tori in G, when the base field F is algebraically closed and of characteristic 0.

In the present study of $W(G)$, we exploit the above results and technique of [3] and, accordingly, we refer to [2] and [3] for standard facts concerning affine algebraic groups and their automorphism group.

The following are brief descriptions of the contents appearing in each section: In §2, we examine reductive affine algebraic groups and their conservativeness and, in §3, we establish a certain $W(G)$-invariant decomposition of G when G is conservative. Finally, in §4, we use the result obtained in §3 to study the structure of $W(G)$.

The following notation is standard throughout: Let G be an affine algebraic group. Then G_1 denotes the connected component of the identity element of G and $Z(G)$ the center of G. If $x \in G$, we use I_x to denote the inner automorphism of G that is induced by x, and, for a subset S of G,

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$\mathrm{Int}_G(S)$ denotes $\{I_x: x \in S\}$. In the case where $S = G$, we simply write $\mathrm{Int}(G)$ instead of $\mathrm{Int}_G(G)$.

2. **Reductive groups and conservativeness.** For an affine algebraic group G over a field F, let $\mathfrak{L}(G)$ denote the Lie algebra of G, and for a morphism ρ: $G \to H$ of affine algebraic groups, $\mathfrak{L}(\rho)$ denotes the Lie algebra homomorphism induced by ρ. Thus $\mathfrak{L}(G)$ consists of all F-linear maps X: $\mathfrak{A}(G) \to F$ such that $X(fg) = X(f)g(1) + f(1)X(g)$ for all $f, g \in \mathfrak{L}(G)$, and the map $\mathfrak{L}(\rho)$: $\mathfrak{L}(G) \to \mathfrak{L}(H)$ is given by $\mathfrak{L}(\rho)(X)(f) = X(f \circ \rho)$, $f \in \mathfrak{A}(G)$ and $X \in \mathfrak{L}(G)$. For $x \in G$ and $f \in \mathfrak{A}(G)$, we write $x \cdot f$ for the left translate of f by x, which is given by $(x \cdot f)(y) = f(yx)$ for $y \in G$ and define x/f: $W(G) \to F$ by $(x/f)(\alpha) = f(\alpha(x))$.

With this preparation, we prove the following characterization of conservative reductive affine algebraic groups.

THEOREM 2.1. *Let G be a reductive affine algebraic group over an algebraically closed field F of characteristic* 0. *Then G is conservative if and only if* $\mathrm{Int}(G)$ *is of finite index in* $W(G)$.

PROOF. Suppose $\mathrm{Int}(G)$ is of finite index in $W(G)$. Then the Hopf algebra $\mathfrak{A}(G)$ is locally finite as an $\mathrm{Int}(G)$-module. Since $\mathrm{Int}(G)$ is a normal subgroup of $W(G)$, it is then locally finite as a $W(G)$-module, proving that G is conservative.

Suppose, conversely, that G is conservative. Thus, by Theorem 2.1, [4], $W(G)$ is an affine algebraic group and its F-algebra $\mathfrak{A}(W(G))$ of polynomial functions on $W(G)$ is generated by the functions x/f, $x \in G$ and $f \in \mathfrak{A}(G)$, and their antipodes.

We first show that the F-space $\mathfrak{L}(W(G))$ may be identified with an F-subspace of the space $Z^1(G, \mathfrak{L}(G))$ of all nonhomogeneous rational 1-cocycles of G with coefficients in $\mathfrak{L}(G)$ relative to the adjoint action of G on $\mathfrak{L}(G)$. To do this, we let $\sigma \in \mathfrak{L}(W(G))$ and, for each $x \in G$, we define

$$\sigma_x: \mathfrak{A}(G) \to F$$

by

$$\sigma_x(f) = \sigma(x/x^{-1} \cdot f), \qquad f \in \mathfrak{A}(G).$$

Then we see easily that $\sigma_X \in \mathfrak{L}(G)$ for all $x \in G$, and we also have

$$\sigma_{xy} = \sigma_x + \mathrm{Ad}(x)(\sigma_y), \qquad x, y \in G. \tag{1}$$

To see this, let γ: $\mathfrak{A}(G) \to \mathfrak{A}(G) \otimes \mathfrak{A}(G)$ be the comultiplication of the Hopf algebra $\mathfrak{A}(G)$. For each $f \in \mathfrak{A}(G)$, we write

$$\gamma(f) = \sum_{i=1}^{n} f_i \otimes g_i, \qquad f_i, g_i \in \mathfrak{A}(G). \tag{2}$$

Then we have

$$f(xy) = \sum_{i=1}^{n} f_i(x)g_i(y) \quad \text{for } x, y \in G. \tag{3}$$

Now let $\alpha \in W(G)$. Then

$$\begin{aligned}
\big(xy/(xy)^{-1}\cdot f\big)(\alpha) &= f\big(\alpha(x)\alpha(y)y^{-1}x^{-1}\big)\\
&= f\big(\alpha(x)x^{-1}\cdot I_x(\alpha(y)y^{-1})\big)\\
&= \sum_{i=1}^{n} f_i\big(\alpha(x)x^{-1}\big)g_i\big(I_x(\alpha(y)y^{-1})\big) \quad \text{(by (3))}\\
&= \sum_{i=1}^{n} (x/x^{-1}\cdot f_i)(\alpha)\big(y/y^{-1}\cdot(g_i\circ I_x)\big)(\alpha).
\end{aligned}$$

That is, we have

$$xy/(xy)^{-1}\cdot f = \sum_{i=1}^{n} (x/x^{-1}\cdot f_i)\cdot\big(y/y^{-1}(g_i\circ I_x)\big). \tag{4}$$

Now

$$\begin{aligned}
\sigma_{xy}(f) = \sigma\big(xy/(xy)^{-1}\cdot f\big) &= \sigma\left(\sum_{i=1}^{n} (x/x^{-1}\cdot f)\cdot\big(y/y^{-1}(g_i\circ I_x)\big)\right)\\
&= \sum_{i=1}^{n} \sigma\big(x/x^{-1}\cdot f_i\big)g_i(1) + \sum_{i=1}^{n} f_i(1)\sigma\big(y/y^{-1}\cdot(g_i\circ I_x)\big).
\end{aligned}$$

However, we have (using (3))

$$x/x^{-1}\cdot f = \sum_{i=1}^{n} (x/x^{-1}\cdot f_i)g_i(1), \quad \text{and}$$

$$y/y^{-1}\cdot(f\circ I_x) = \sum_{i=1}^{n} \big(y/y^{-1}\cdot(g_i\circ I_x)\big)f_i(1)$$

Hence

$$\begin{aligned}
\sigma_{xy}(f) &= \sigma(x/x^{-1}\cdot f) + \sigma\big(y/y^{-1}\cdot(f\circ I_x)\big) = \sigma_x(f) + \sigma_y(f\circ I_x)\\
&= (\sigma_x + \mathrm{Ad}(x)(\sigma_y))(f),
\end{aligned}$$

proving (1).

For each $\sigma \in \mathfrak{L}(W(G))$, define $\sigma'\colon G \to \mathfrak{L}(G)$ by $\sigma'(x) = \sigma_x$, $x \in G$. Then we easily see that $\sigma' \in Z^1(G, \mathfrak{L}(G))$. Since the functions x/f, together with their antipodes, generate $\mathcal{Q}(W(G))$ as an F-algebra, it follows that the F-linear map $\sigma \to \sigma'$ is injective, under which we identify $\mathfrak{L}(W(G))$ with an F-subspace of $Z^1(G, \mathfrak{L}(G))$.

We next consider the morphism of affine algebraic groups $\nu\colon G \to W(G)$, which is given by $\nu(x) = I_x$, $x \in G$.

We compute the image of $\mathfrak{L}(G)$ under the F-linear map $\mathfrak{L}(\nu)\colon \mathfrak{L}(G) \to$

$\mathfrak{L}(W(G))$, $\mathfrak{L}(W(G))$ being identified with an F-subspace of $Z^1(G, \mathfrak{L}(G))$.

To do this, we first note that $X(f') = -X(f)$ for all $f \in \mathcal{Q}(G)$ and $X \in \mathfrak{L}(G)$. This may be seen as follows: Write $\gamma(f) = \sum_{i=1}^n f_i \otimes g_i$ as in (2). Then, by (3),

$$f(1) = f(xx^{-1}) = \sum_{i=1}^n f_i(x)g_i'(x) = \left(\sum_{i=1}^n f_i g_i'\right)(x),$$

which implies that $\sum_{i=1}^n f_i g_i'$ is constant.

Hence

$$\begin{aligned} 0 = X\left(\sum_{i=1}^n f_i g_i'\right) &= \sum_{i=1}^n X(f_i)g_i'(1) + \sum_{i=1}^n f_i(1)X(g_i') \\ &= X\left(\sum_{i=1}^n f_i g_i(1)\right) + X\left(\sum f_i'(1)g_i'\right) \\ &= X(f) + X(f') \end{aligned}$$

and $X(f') = -X(f)$ follows.

For $X \in \mathfrak{L}(G)$, $x \in G$, and $f \in \mathcal{Q}(G)$, we have

$$\mathfrak{L}(\nu)(X)(x)(f) = \mathfrak{L}(\nu)(X)(x/x^{-1}\cdot f) = X\big((x/x^{-1}\ f)\cdot \nu\big).$$

But $(x/x^{-1}\cdot f)\cdot\nu = \sum_{i=1}^n f_i \cdot (g_i\cdot\nu(x))'$.

Hence

$$\begin{aligned} \mathfrak{L}(\nu)(X)(x)(f) &= X\left(\sum_{i=1}^n f_i\cdot(g_i\cdot\nu(x))'\right) \\ &= \sum_{i=1}^n X(f_i)(g_i\cdot\nu(x))'(1) + \sum_{i=1}^n f_i(1)X(g_i\cdot\nu(x)') \\ &= X\left(\sum_{i=1}^n f_i g_i(1)\right) - X\left(\sum_{i=1}^n f_i(1)(g_i\cdot\nu(x))\right) \\ &= X(f) - X(f\cdot\nu(x)) = (X - \mathrm{Ad}(x)(X))(f). \end{aligned}$$

That is, $\mathfrak{L}(\nu)(X)(x) = X - \mathrm{Ad}(x)(X)$, and we see that $\mathrm{Im}(\mathfrak{L}(\nu))$ is equal to the subsapce $B^1(G, \mathfrak{L}(G))$ of $Z^1(G, \mathfrak{L}(G))$ consisting of all 1-coboundaries of G.

Since G is reductive, $H^1(G, \mathfrak{L}(G)) = 0$. Hence $\mathrm{Im}(\mathfrak{L}(\nu)) = B^1(G, \mathfrak{L}(G)) = Z^1(G, \mathfrak{L}(G))$. Since F is algebraically closed, the surjectivity of $\mathfrak{L}(\nu)$ implies that $\mathrm{Im}(\nu) = \mathrm{Int}(G)$ is open in $W(G)$ and hence $\mathrm{Int}(G)$ is of finite index in $W(G)$.

THEOREM 3.2. *Let G be an affine algebraic group over an algebraically closed* *field F of characteristic 0. Then G is conservative if a maximal reductive subgroup of G is conservative.*

PROOF. Let G_u denote the unipotent radical of G, and let P be a maximal reductive subgroup of G. Since F is of characteristic 0, a theorem of Mostow (see [2, Theorem 14.2]) assures that we have a semidirect product decomposition $G = G_u \cdot P$. By the conjugacy of maximal reductive subgroups, we may assume that P is conservative, and we have $W(G) = \mathrm{Int}(G) \cdot \mathscr{A}$, where $\mathscr{A}$ is the subgroup of $W(G)$ consisting of all $\alpha \in W(G)$ leaving P invariant.

Let $\mathscr{A}_P$ denote the restriction image of $\mathscr{A}$ in $W(P)$. Then $\mathrm{Int}(P) < \mathscr{A}_p$, and, since P is conservative, $W(P)/\mathrm{Int}(P)$ is finite by Theorem 2.1. It follows that $\mathscr{A}_p/\mathrm{Int}(P)$ is also finite.

From this point on, we can copy the argument used in [3, p. 539] for the proof of conservativeness of G when P is a connected semi-simple algebraic subgroup and conclude that G is conservative. This establishes Theorem 2.2.

3. **$W(G)$-invariant decomposition of** G. For a subset $\mathscr{A}$ of $W(G)$, let $G^{\mathscr{A}}$ denote the set consisting of all $x \in G$ such that $\alpha(x) = x$ for all $\alpha \in \mathscr{A}$.

We prove the following result which will then be used in §4 for out study of $W(G)$.

THEOREM 3.1. *Let G be a connected conservative affine algebraic group over an algebraically closed field F of characteristic 0, and let T be the maximal central torus of $W(G)_1$. Then there exists a $W(G)$-invariant algebraic vector subgroup Z of G such that $G = Z \times G^T$.*

PROOF. If T is trivial, then the assertion holds trivially. Thus we assume that T is of dimension $\geqslant 1$.

For each $x \in G$, the inner automorphism I_x induced by x commutes with every element of T. Hence, for $\alpha \in T$ and $x \in G$, we have $x^{-1}\alpha(x) \in Z(G)$.

We define, for each $\alpha \in T$, $\eta_\alpha\colon G \to Z(G)$ by $\eta_\alpha(x) = x^{-1}\alpha(x)$, $x \in G$.

Then η_α is a morphism of affine algebraic groups. Since G is connected, it follows that $\eta_\alpha(x) \in Z(G)_1$ for all $x \in G$. Now we choose a maximal reductive subgroup P of G so that $G = G_u \cdot P$ (semidirect). We first show that every element of P is T-fixed. To do this, we choose a maximal torus D of P. Then $P = D \cdot P'$, where P' denotes the commutator subgroup of P, and $P' < \mathrm{Ker}\, \eta_\alpha$ implies that every element of P' is T-fixed. Hence it is enough to show that every element of D is T-fixed. Let K be the maximal torus of $Z(G)$. Then the torus $\eta_\alpha(D)$ is contained in K, and hence we see that every element α of T leaves D invariant. Consider the polynomial map

$$\phi\colon T \times D \to D,$$

given by $\phi(\alpha, x) = \alpha(x)$, and define, for each $x \in D$, $\phi_x\colon T \to D$ by $\phi_x(\alpha) = \alpha(x)$. Then clearly ϕ_x is a polynomial map. Let $x \in D$ be of order $m < \infty$.

Then $\phi_x(\alpha)$ is also of order m for all $\alpha \in T$. Since D contains only a finite number of elements of order m, it follows from the connectedness of T that $\operatorname{Im}\phi_x = \{x\}$. That is, $\alpha(x) = x$ for all $\alpha \in T$. Since the elements in D of finite order form a dense subset of D, it follows that T leaves every element of D fixed.

Next we show that if U denotes the unipotent radical of $Z(G)$, then $G = U \cdot G^T$. The morphism $\eta_\alpha: G \to Z(G)$ for $\alpha \in T$ maps G_u into U. Hence η_α induces a morphism $\mu_\alpha: G_u \to U$ of affine algebraic groups. Let μ_α^0 denote $\mathfrak{L}(\mu_\alpha): \mathfrak{L}(G_u) \to \mathfrak{L}(U)$. The natural action of T on U determines a T-module structure on the F-space $\mathfrak{L}(U)$, and this in turn defines a T-module structure on the F-space $\operatorname{Hom}_F(\mathfrak{L}(G_u), \mathfrak{L}(U))$.

We then have

$$\mu_{\beta}^0 = \mu_\beta^0 + \alpha \cdot \mu_\beta^0, \qquad \alpha, \beta \in T. \tag{1}$$

To prove (1), we note that $\exp_U \cdot \mu_\alpha^0 = \mu_\alpha \cdot \exp_{G_u}$, where $\exp_U$, $\exp_{G_u}$ denote the exponential maps for U, G_u, respectively. Hence for $X \in \mathfrak{L}(G_u)$,

$$\begin{aligned}
\exp \mu_{\alpha\beta}^0(X) &= \mu_{\alpha\beta}(\exp X) = (\exp X)^{-1}\alpha\beta(\exp X)\\
&= (\exp X)^{-1}\alpha(\exp X)\alpha\big((\exp X)^{-1}\beta(\exp X)\big)\\
&= \mu_\alpha(\exp X)\alpha(\mu_\beta(\exp X)) = \exp \mu_\alpha^0(X)\alpha\big(\exp \mu_\beta^0(X)\big)\\
&= \exp \mu_\alpha^0(X)\exp\big(\mathfrak{L}(\alpha)(\mu_\beta^0(X))\big) = \exp\big(\mu_\alpha^0(X) + \alpha \cdot \mu_\beta^0(X)\big).
\end{aligned}$$

Hence it follows that $\mu_{\alpha\beta}^0(X) = \mu_\alpha^0(X) + \alpha \cdot \mu_\beta^0(X)$, proving (1).

The identity (1) defines a rational T-module structure on the F-space $F \oplus \operatorname{Hom}_F(\mathfrak{L}(G_u), U)$, if we define the T-action by $\alpha \cdot (r, \phi) = (r, r\mu_\alpha^0 + \alpha \cdot \phi)$ for $\alpha \in T$, $r \in F$ and $\phi \in \operatorname{Hom}_F(\mathfrak{L}(G_u), \mathfrak{L}(U))$. Since T is reductive, the T-submodule $\operatorname{Hom}_F(\mathfrak{L}(G_u), \mathfrak{L}(U))$ has a 1-dimensional T-invariant complement in $F \oplus \operatorname{Hom}_F(\mathfrak{L}(G_u), \mathfrak{L}(U))$. This complement contains exactly one element of the form $(1, \phi)$.

Hence $(1, \phi) = \alpha \cdot (1, \phi) = (1, \mu_\alpha^0 + \alpha \cdot \phi)$ for all $\alpha \in T$ and this implies that $\mu_\alpha^0 = \phi - \alpha \cdot \phi$, $\alpha \in T$.

For each $X \in \mathfrak{L}(G_u)$, we have

$$\begin{aligned}
\exp \phi(X) &= \exp\big(\mu_\alpha^0(X) + \alpha \cdot \phi(X)\big)\\
&= \exp\big(\mu_\alpha^0(X)\big)\exp\big(\mathfrak{L}(\alpha)(\phi(X))\big)\\
&= (\exp X)^{-1}\alpha(\exp X)\alpha(\exp \phi(X)).
\end{aligned}$$

Hence $\exp_{G_u} X \cdot \exp_U \phi(X) \in G^T$ for all $X \in \mathfrak{L}(G_u)$. Since $\exp_{G_u}(\mathfrak{L}(G_u)) = G_u$, it follows that $G_u < U \cdot G^T$, and $p < G^T$ implies $G = U \cdot G^T$.

Now we consider the rational T-module $\mathfrak{L}(U)$. Since T is a torus over an algebraically closed field, we may decompose the F-space $\mathfrak{L}(U)$ as

$$\mathfrak{L}(U) = \sum_{\chi \neq 1} L_\chi + \mathfrak{L}(U)^T,$$

where L_χ is the weight space $\{X \in \mathfrak{L}(U)\colon \alpha \cdot X = \chi(\alpha)X \text{ for all } \alpha \in T\}$ corresponding to the weight $\chi\colon T \to F^*$, and $\mathfrak{L}(U)^T$ is the T-fixed part of $\mathfrak{L}(U)$.

Since T is a normal subgroup of $W(G)$, $W(G)$ permutes the weights of T in $\mathfrak{L}(U)$. Hence the F-subspace $Z = \Sigma_{\chi = 1} L_\chi$ is $W(G)$-invariant. Let $Z = \exp_U Z$. Then $U = Z \times U^T$ and this implies that $G = Z \times G^T$ follows. Clearly Z is $W(G)$-invariant and the theorem is proved.

REMARK. Since T is a normal subgroup of $W(G)$, it follows that G^T is also $W(G)$-invariant. As we will see in §4, T is central in $W(G)$ and, in fact, a direct factor of $W(G)$.

4. Decomposition and conservativeness of $W(G)$.

THEOREM 4.1. *Let G be a conservative connected affine algebraic group over an algebraically closed field F of characteristic* 0. *Then the maximal central torus of $W(G)_1$ is of dimension $\leqslant 1$ and is a direct factor of $W(G)$.*

PROOF. Let T be the maximal central torus of $W(G)_1$, and assume that T is nontrivial. Then we have a $W(G)$-invariant decomposition $G = Z \times G^T$ (Theorem 3.1). Hence we have $W(G) \simeq W(Z) \times W(G^T)$ as affine algebraic groups and the restriction map $T \to W(Z)$ is injective.

Let $\mathfrak{z}$ denote the Lie algebra of Z. Then the affine algebraic group $W(Z)$ may be identified with the affine algebraic group $GL(\mathfrak{z})$ of all F-linear automorphisms of $\mathfrak{z}$. Since F is algebraically closed, the center of $W(Z)$ is a 1-dimensional torus and is a direct factor of $W(Z)$. Since every element of $W(Z)$ can be extended to an element of $W(G)$, we see easily that the restriction map sends T isomorphically onto the center of $W(Z)$. Hence our assertion follows.

In [2], Hochschild proved that, if G is a nonabelian unipotent affine algebraic group, then the maximal central torus of $W(G)_1$ is trivial and hence that $W(G)_1$ is conservative. The assertion does not hold for arbitrary solvable affine algebraic groups (see the example in [2, p. 111]).

The following theorem characterizes those nonabelian solvable groups G for which $W(G)_1$ is conservative.

THEOREM 4.2. *Let G be a connected conservative solvable nonabelian affine algebraic group over an algebraically closed field of characteristic* 0. *Then the following are equivalent*:

(i) *$W(G)_1$ is conservative.*

(ii) *The connected component of the center of $W(G)_1$ is unipotent* (*i.e.* $T = 1$).

(iii) *G cannot be a product $G = Z \times H$ of a nontrivial algebraic vector subgroup Z and an algebraic subgroup H, both of which are invariant under $W(G)$.*

PROOF. (iii) $\rightarrow$ (ii) follows from Theorem 3.1 and the subsequent remark.

(ii) $\rightarrow$ (iii) holds because of the decomposition $W(G) = W(Z) \times W(H)$.

(ii) $\rightarrow$ (i) follows from Theorem 3.2 of [4].

It remains to show (i) $\rightarrow$ (ii).

Let K be a maximal torus of G so that $G = G_u \cdot K$ (semidirect).

If K is trivial, then G is unipotent and nonabelian, and hence (ii) holds (see [2, p. 110]).

(1) Suppose $\dim K \geqslant 2$. Then the maximal central torus of G is trivial by Theorem 3.2 [4] and this implies that the torus $\mathrm{Int}_G(K) \simeq KZ(G)/Z(G)$ is of dimension $\geqslant 1$. Since $\mathrm{Int}(G)$ is a normal algebraic subgroup of $W(G)$, it follows that the algebraic torus $\mathrm{Int}_G(K)$ is contained in the radical of $W(G)_1$ and hence is central in a maximal reductive group containing it. Since $W(G)_1$ is conservative, (ii) follows from Theorem 3.2 of [3].

(2) Suppose $\dim K = 1$. If K is central in G, then $G = G_u \times K$, and hence $W(G) \simeq W(G_u) \times \mathbf{Z}_2$. Since G_u is nonabelian, (ii) follows immediately.

Therefore we may assume that the identity component of the center of G is unipotent. Then $\mathrm{Int}_G(K)$ is a 1-dimensional torus. Assume that (ii) does not hold, and let T be the maximal central torus of $W(G)_1$. Then $T \cap \mathrm{Int}_G(K) = \{1\}$, for if $\alpha \in T$ is of the form $\alpha = I_x$ for some $x \in K$, then the decomposition $G = Z \times G^T$ in Theorem 3.1 implies that $\alpha = 1$.

Since T centralizes $\mathrm{Int}_G(K)$, it follows that $T' = T \cdot \mathrm{Int}_G(K)$ (direct) is an algebraic torus of dimension 2.

Since T' is contained in the radical of $W(G)_1$, it follows that T' is central in a maximal reductive subgroup containing T'. (See [1, Chapter III].) Hence again by Theorem 3.2 of [3], $W(G)_1$ cannot be conservative, contradicting (i). Therefore $T = \{1\}$ and (ii) is proved.

REFERENCES

1. A. Borel, *Linear algebraic groups*, Benjamin, New York, 1969. MR **40** #4273.

2. G. Hochschild, *Introduction to affine algebraic groups*, Holden-Day, San Francisco, 1971. MR **43** #3268.

3. G. Hochschild and G. D. Mostow, *Automorphisms of affine algebraic groups*, J. Algebra **22** (1972), 535–543.

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